

# The sum of the length minus one of extremal rays on Fano manifolds

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## Abstract

Let  $X$  be a complex Fano manifold of dimension  $n$ . Let  $s(X)$  be the sum of  $l(R) - 1$  for all the extremal rays of  $X$ , the edges of the cone  $\text{NE}(X)$  of curves of  $X$ , where  $l(R)$  denotes the minimum of  $(-K_X \cdot C)$  for all rational curves  $C$  whose class  $[C]$  belongs to  $R$ . We show  $s(X) \leq n$  if  $n \leq 4$ . And for  $n \leq 4$ , we completely classify the case the equality holds.

## 1 Introduction

Let  $X$  be an arbitrary Fano manifold of dimension  $n$  and Picard number  $\rho_X$ . In 1988, Mukai [Muk88] made the following conjecture.

**Conjecture 1.1.** *One has*

$$\rho_X(r_X - 1) \leq n,$$

*and the equality holds if and only if  $X \simeq (\mathbb{P}^{r_X-1})^{\rho_X}$ , where*

$$r_X := \max\{m \in \mathbb{N} \mid -K_X \sim mL \text{ for some Cartier divisor } L\}.$$

Wiśniewski [Wiś90] strengthened it into the following form to prove Conjecture 1.1.

**Conjecture 1.2.** *One has*

$$\rho_X(\iota_X - 1) \leq n,$$

*and the equality holds if and only if  $X \simeq (\mathbb{P}^{\iota_X-1})^{\rho_X}$ , where*

$$\iota_X := \min\{(-K_X \cdot C) \mid C \subset X \text{ is a rational curve}\}.$$

Conjecture 1.2 has been proved for  $n \leq 5$  by Andreatta, Chierici, and Occhetta [ACO04]; in the toric case by Casagrande [Cas06]; and for  $\iota_X \geq (n+3)/3$  by Novelli and Occhetta [NO10].

Recently, Tsukioka [Tsu10c] generalized Conjecture 1.2 as follows.

**Conjecture 1.3.** *One has*

$$\rho_X(l_X - 1) \leq n,$$

*and the equality holds if and only if  $X \simeq (\mathbb{P}^{l_X-1})^{\rho_X}$ , where  $l_X$  denotes the minimum of the length  $l(R)$  of all the extremal rays  $R$  of  $X$ , and*

$$l(R) := \min\{(-K_X \cdot C) \mid C \subset X \text{ is a rational curve with } [C] \in R\}.$$

We think that it is more natural to consider *all* the extremal rays to study a Fano manifold. We set up the following question.

**Question 1.4.** *Give a bound of*

$$s(X) := \sum_{R \in \text{NE}(X) \text{ extremal ray}} (l(R) - 1)$$

*for arbitrary Fano manifolds  $X$  with dimension  $n$ .*

In this paper, we identify the bound in the case  $n \leq 4$ .

**Theorem 1.5** (Main Theorem). *Let  $X$  be a Fano manifold of dimension  $n$ .*

(i) *Let  $n \leq 3$ . Then  $s(X) \leq n$ , and equality holds if and only if*

$$X \simeq \prod_{R \in \text{NE}(X) \text{ extremal ray}} \mathbb{P}^{l(R)-1}.$$

(ii) *Let  $n = 4$ . Then  $s(X) \leq n$ , and equality holds if and only if*

$$X \simeq \prod_{R \in \text{NE}(X) \text{ extremal ray}} \mathbb{P}^{l(R)-1}$$

*or*

$$X \simeq \text{Bl}_{p,q}(\mathbb{Q}^4),$$

*where  $\mathbb{Q}^4 \subset \mathbb{P}^5$  be a smooth hyperquadric and  $p, q \in \mathbb{Q}^4$  distinct points with  $\overline{pq} \not\subset \mathbb{Q}^4$ , where  $\overline{pq} \subset \mathbb{P}^5$  denotes the line through  $p, q$ .*

**Remark 1.6.** If  $n \geq 5$ , then there exists a Fano manifold  $X$  of dimension  $n$  with  $s(X) > n$  (see Remark 3.5 (iii)).

As an immediate consequence of Theorem 1.5, we can give the affirmative answer to Conjecture 1.3 in the case  $n \leq 4$  (Tsukioka [Tsu10c] proved the inequality in the case  $n = 4$  but did not settle the assertion on the equality case).

**Corollary 1.7** (cf. [Tsu10c]). *Conjecture 1.3 is true for  $n \leq 4$ .*

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**Notation and terminology.** We always work over the complex number field  $\mathbb{C}$ . For a normal projective variety  $X$ , we denote the normalization of the space of irreducible and reduced rational curves on  $X$  by  $\text{RatCurves}^n(X)$  (see [Kol96, Definition II.2.11]). For the theory of extremal contraction, we refer the readers to [KM98].

For a projective klt pair  $(X, \Delta)$  with effective  $\Delta$  and a  $(K_X + \Delta)$ -negative extremal ray  $R \subset \overline{\text{NE}}(X)$ , we say that  $R$  is *of fiber type* (resp. *divisorial*, *small*) if the associated contraction morphism  $\text{cont}_R: X \rightarrow Y$  is of fiber type (resp. divisorial, small). We define the exceptional locus of  $R$  by  $\text{Exc}(R) := \{x \in X \mid \text{cont}_R: X \rightarrow Y \text{ is not isomorphism at } x\}$ . For example, if  $R$  is of fiber type, then  $\text{Exc}(R) = X$ .

For a smooth projective variety  $X$  and a  $K_X$ -negative extremal ray  $R \subset \overline{\text{NE}}(X)$ , we say  $R$  is *of type*  $(a, b)$  if  $\dim \text{Exc}(R) = a$  and  $\dim \text{cont}_R(\text{Exc}(R)) = b$ , and we say  $R$  is *of type*  $(n-1, b)^{\text{sm}}$  if the associated contraction morphism is the blowing up of a smooth projective variety along a smooth subvariety of dimension  $b$ . We define the *length*  $l(R)$  of  $R$  by  $l(R) := \min\{(-K_X \cdot C) \mid C \text{ is a rational curve with } [C] \in R\}$ .

For an algebraic variety  $X$  and a closed subscheme  $Y \subset X$ ,  $\text{Bl}_Y(X) \rightarrow X$  denotes the blowing up of  $X$  along  $Y$ . The symbol  $\mathbb{Q}^n$  denotes a smooth hyperquadric in  $\mathbb{P}^{n+1}$ . We say that  $X$  is a *Fano manifold* if  $X$  is a smooth projective variety whose anticanonical divisor  $-K_X$  is ample.

## 2 Preliminaries

### A family of rational curves and extremal contractions.

**Definition 2.1** (see for example [ACO04]). Let  $X$  be a normal projective variety.

We define a *family of rational curves* to be an irreducible component  $H \subset \text{RatCurves}^n(X)$  with the induced universal family. For any  $x \in X$ , let  $H_x$  be the subvariety of  $H$  parametrizing rational curves passing through  $x$ , and  $\tilde{H}_x$  the normalization of the image of  $H_x$  in  $\text{Chow}(X)$ . We define  $\text{Locus}(H)$  (resp.  $\text{Locus}(H_x)$ ) to be the union of rational curves parametrized by  $H$  (resp.  $H_x$ ). Similarly, for any subset  $Y \subset X$ , let

$$\text{Locus}(H)_Y := \bigcup_{[C] \in H; Y \cap C \neq \emptyset} C.$$

For a family  $H$  of rational curves on  $X$ ,  $H$  is said to be *dominating* if  $\overline{\text{Locus}(H)} = X$ , *unsplit* if  $H$  is projective, and *locally unsplit* if  $H_x$  is projective for general  $x \in \text{Locus}(H)$ .

**Proposition 2.2** ([NO10, Proposition 2.5(b)]). *Let  $X$  be a smooth projective variety,  $H$  a family of rational curves on  $X$ , and  $x \in \text{Locus}(H)$  a point such that  $H_x$  is projective. Then one has*

$$\dim \text{Locus}(H) + \dim \text{Locus}(H_x) \geq \dim X + (-K_X \cdot \text{Fam } H) - 1,$$

where  $\text{Fam } H$  is the numerical class of the curves in  $X$  parametrized by  $H$ .

**Lemma 2.3** ([ACO04, Lemma 5.4(b)]). *Let  $X$  be a smooth projective variety,  $Y \subset X$  be an irreducible closed subset, and  $H$  be an unsplit family of rational curves on  $X$ . We assume that all curves which contained in  $Y$  are numerically independent from curves parametrized by  $H$ , and that  $Y \cap \text{Locus}(H) \neq \emptyset$ . Then one has*

$$\dim \text{Locus}(H)_Y \geq \dim Y + (-K_X \cdot \text{Fam } H) - 1.$$

**Lemma 2.4** ([Occ06, Lemma 3.2]). *Let  $X$  be a smooth projective variety,  $Y \subset X$  be a closed subset, and  $H$  be an unsplit family of rational curves on  $X$ . Then every curve contained in  $\text{Locus}(H)_Y$  is numerically equivalent to a linear combination with rational coefficients of curves in  $Y$  and curves parametrized by  $Y$ .*

**Proposition 2.5.** *Let  $X$  be a smooth projective variety of dimension  $n$ . Assume that there exist distinct  $K_X$ -negative extremal rays  $R_1, R_2 \subset \overline{\text{NE}}(X)$  such that*

- (i)  $R_1$  is of type  $(n-1, 0)$ ,
- (ii)  $l(R_2) \geq 2$ ,
- (iii)  $\text{Exc}(R_1) \cap \text{Exc}(R_2) \neq \emptyset$ .

Then  $R_2$  is of fiber type and  $\rho_X = 2$ .

*Proof.* Let  $E_i := \text{Exc}(R_i)$  for  $i = 1, 2$  and fix  $x \in E_1 \cap E_2$ . Let  $C \subset X$  be a rational curve such that

- (1)  $x \in C$  and  $[C] \in R_2$ ,
- (2)  $(-K_X \cdot C)$  is minimal among satisfying (1).

Let  $H$  be a family of rational curves containing  $[C] \in \text{RatCurves}^n(X)$ . Then  $H_x$  is projective by construction. If there exists an irreducible curve  $l \subset E_1 \cap \text{Locus}(H_x)$  then  $[l] \in R_1 \cap R_2 = \{0\}$ , which leads to a contradiction. Hence  $\dim(E_1 \cap \text{Locus}(H_x)) = 0$ . Thus  $\dim \text{Locus}(H_x) \leq 1$  since  $\dim E_1 = n-1$ . Therefore

$$\begin{aligned} 1 &\geq \dim \text{Locus}(H_x) \geq (n - \dim \text{Locus}(H)) + (-K_X \cdot \text{Fam } H) - 1 \\ &\geq l(R_2) - 1 \geq 1 \end{aligned}$$

holds by Proposition 2.2. Thus  $\dim \text{Locus}(H) = n$  and  $l(R_2) = (-K_X \cdot \text{Fam } H) = 2$ . In particular,  $H$  is dominating and unsplit. We have

$$\dim \text{Locus}(H)_{E_1} \geq \dim E_1 + (-K_X \cdot \text{Fam } H) - 1 = n$$

by Lemma 2.3. Therefore  $\text{Locus}(H)_{E_1} = X$  and  $N_1(X)$  is spanned by  $\mathbb{R}R_1$  and  $\mathbb{R}R_2$  by Lemma 2.4.  $\square$

**Theorem 2.6** ([Wi91, Theorem 1.1]). *Let  $X$  be a smooth projective variety and  $R \in \overline{\text{NE}}(X)$  be a  $K_X$ -negative extremal ray. Then for every irreducible component  $E \subset \text{Exc}(R)$ , we have*

$$l(R) \leq \dim X + 1 - 2\text{codim}_X E - \dim \text{contr}_R(E).$$

**Remark 2.7.** Let  $(X, \Delta)$  be a projective klt pair of dimension  $n$  with effective  $\mathbb{Q}$ -divisor  $\Delta$ .

- (1) If  $\rho_X \geq 3$  then no  $(K_X + \Delta)$ -negative extremal rays are of type  $(n, 0)$  or of type  $(n, 1)$ .
- (2) Set  $m \geq 2$ . Let  $R_i \subset \overline{\text{NE}}(X)$  be a  $(K_X + \Delta)$ -negative extremal ray, the associated contraction  $\varphi_i: X \rightarrow Z_i$ ,  $E_i := \text{Exc}(R_i)$ , and  $C_i \subset X$  an irreducible curve with  $[C_i] \in R_i$  for any  $1 \leq i \leq m$ . We assume that  $E_i \cap E_j = \emptyset$  for any  $1 \leq i < j \leq m$ . Then we can define the morphism  $\varphi: X \rightarrow Z$  contracting all of  $E_1, \dots, E_m$  (Glue  $\varphi_1, \dots, \varphi_m$  together. We note that  $Z$  is a normal proper variety). Then there is an exact sequence of the form

$$0 \longrightarrow \text{Pic } Z \xrightarrow{\varphi^*} \text{Pic } X \xrightarrow{((\bullet \cdot C_1), \dots, (\bullet \cdot C_m))} \mathbb{Z}^{\oplus m}.$$

Furthermore, if  $X$  is  $\mathbb{Q}$ -factorial and  $R_i$  is divisorial for any  $1 \leq i \leq m$ , then  $Z$  is also  $\mathbb{Q}$ -factorial and hence  $\rho_Z \geq 1$ .

*Proof.* The assertion (1) is obvious. We prove (2).

For  $1 \leq i \leq m$ , let  $\psi_i: X \rightarrow Z_i$  be the morphism contracting  $E_1, \dots, E_i$ . (Glue  $\varphi_1, \dots, \varphi_i$  together. We note that  $Z_i$  is also a normal proper variety.) We also let  $\pi_i: Z_{i-1} \rightarrow Z_i$  be the induced morphism with respect to  $\psi_{i-1}$  and  $\psi_i$  (contracting  $E_i$ ) such that  $\pi_i \circ \psi_{i-1} = \psi_i$ . We denote  $Z_0 := X$  and  $\pi_1 := \varphi_1 = \psi_1$ . We note that  $\varphi = \psi_m$ .

It is enough to show the exactness of

$$0 \longrightarrow \text{Pic } Z_i \xrightarrow{\pi_i^*} \text{Pic } Z_{i-1} \xrightarrow{(\bullet \cdot C_i)} \mathbb{Z}$$

for any  $1 \leq i \leq m$  to prove (2).

We can assume  $2 \leq i \leq m$  since the case  $i = 1$  is obvious by the contraction theorem. Injectivity of  $\pi_i^*: \text{Pic } Z_i \rightarrow \text{Pic } Z_{i-1}$  is obvious since the morphism

$\pi_i$  satisfies  $\pi_{i*}\mathcal{O}_{Z_{i-1}} = \mathcal{O}_{Z_i}$ . Let  $\tau_i: Y_i \rightarrow Z_i$  be the morphism contracting  $E_1, \dots, E_{i-1}$  which satisfies the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\psi_{i-1}} & Z_{i-1} \\ \varphi_i \downarrow & & \downarrow \pi_i \\ Y_i & \xrightarrow{\tau_i} & Z_i. \end{array}$$

Let  $V_i := Z_i \setminus (\tau_i \circ \varphi_i(E_1 \sqcup \dots \sqcup E_{i-1}))$  and  $U_i := Z_i \setminus (\tau_i \circ \varphi_i(E_i))$  be open subvarieties of  $Z_i$ .

We choose an invertible sheaf  $M \in \text{Pic } Z_{i-1}$  satisfying  $(M \cdot C_i) = 0$ . Then  $0 = (M \cdot C_i) = (\psi_{i-1}^* M \cdot C_i)$ . Hence there exists an invertible sheaf  $L_1 \in \text{Pic } Y_i$  such that  $\varphi_i^* L_1 \simeq \psi_{i-1}^* M$  holds by the contraction theorem for  $R_i$ . Thus we obtain the isomorphisms

$$M \simeq \psi_{i-1*} \psi_{i-1}^* M \simeq \psi_{i-1*} \varphi_i^* L_1 \simeq \pi_i^* \tau_{i*} L_1$$

since  $\varphi_i$  and  $\pi_i$  are isomorphisms over  $V_i$ , and  $\psi_{i-1}$  and  $\tau_i$  are isomorphisms over  $U_i$ , respectively. We note that  $\tau_{i*} L_1$  is an invertible sheaf since  $\tau_{i*} L_1|_{V_i} \simeq M|_{\pi_i^{-1}(V_i)}$  and  $\tau_{i*} L_1|_{U_i} \simeq L_1|_{\tau_i^{-1}(U_i)}$ . Therefore we have  $M \in \pi_i^*(\text{Pic } Z_i)$ .

We prove the  $\mathbb{Q}$ -factoriality of  $Z_i$  by induction under the assumptions. We assume that  $Z_{i-1}$  is  $\mathbb{Q}$ -factorial. Let us pick an arbitrary prime divisor  $B \subset Z_i$ . Then there exists  $s \in \mathbb{Q}$  such that  $(\pi_*^{-1} B + s E_i \cdot C_i) = 0$  since  $(E_i \cdot C_i) < 0$ . There exists  $L \in \text{Pic } Z_i$  and a positive integer  $m$  such that  $\mathcal{O}_{Z_{i-1}}(m(\pi_*^{-1} B + s E_i)) \simeq \pi_i^* L$  since  $Z_{i-1}$  is  $\mathbb{Q}$ -factorial. Therefore we have  $\mathcal{O}_{Z_i}(mB) \simeq L$ .  $\square$

### Characterizations of products of projective spaces.

**Theorem 2.8** (finiteness of tangent morphism [Keb02, Theorem 2.16]). *Let  $X$  be a normal projective variety and  $H$  a dominating and locally unsplit family of rational curves on  $X$ . For general  $x \in X$ , consider the rational map*

$$\tilde{H}_x \dashrightarrow \mathbb{P}(T_X|_x^\vee)$$

*defined by*

$$[l] \mapsto \mathbb{P}(T_l|_x^\vee).$$

*Then the rational map  $\tau_x$  is a finite morphism.*

**Definition 2.9** (Variety of Minimal Rational Tangents). Under the assumption in Theorem 2.8,  $\tau_x$  is called the *tangent morphism*; its image  $\mathcal{C}_x := \tau_x(\tilde{H}_x) \subset \mathbb{P}(T_X|_x^\vee)$  the *variety of minimal rational tangents*, or shortly *VMRT*, of  $H$  at  $x$ .

Araujo [Ara06] showed a criterion of varieties being products of projective spaces using the method of VMRT.

**Theorem 2.10** ([Ara06, Theorem 1.3]). *Let  $X$  be a smooth projective variety of dimension  $n$  with  $k$  distinct dominating and unsplit family of rational curves  $H_1, \dots, H_k$ . Suppose for a general  $x \in X$ , the associated VMRT of  $H_i$  at  $x$  are linear subspaces of dimension  $\lambda_i - 1$  in  $\mathbb{P}(T_X|_x^\vee)$  such that  $\sum_{i=1}^k \lambda_i = n$ . Then  $X \simeq \prod_{i=1}^k \mathbb{P}^{\lambda_i}$ .*

We give another characterization of products of projective spaces in terms of length of extremal rays.

**Theorem 2.11.** *Let  $X$  be a smooth projective variety of dimension  $n = \sum_{i=1}^k d_i$  with  $d_1, \dots, d_k \in \mathbb{N}$ . Assume that there exist distinct  $K_X$ -negative extremal rays  $R_1, \dots, R_k \subset \overline{\text{NE}}(X)$  such that  $R_i$  are of fiber type with  $l(R_i) \geq d_i + 1$  for all  $1 \leq i \leq k$ . Then  $X \simeq \prod_{i=1}^k \mathbb{P}^{d_i}$ .*

*Proof.* We denote  $\varphi_i := \text{cont}_{R_i}: X \rightarrow Y_i$  and  $e_i := \dim X - \dim Y_i$ . We have  $\sum_{i=1}^k e_i \leq n$  by [Wiś91, Theorem 2.2] and  $e_i \geq l(R_i) - 1$  by Theorem 2.6 for any  $i$ . Hence we obtain the inequality

$$n \geq \sum_{i=1}^k e_i \geq \sum_{i=1}^k (l(R_i) - 1) \geq \sum_{i=1}^k d_i = n.$$

Therefore  $e_i = l(R_i) - 1 = d_i$  holds.

Let  $F_i$  be a general fiber of  $\varphi_i$ . Then  $F_i$  is a Fano manifold of dimension  $d_i$  and  $\iota_{F_i} \geq d_i + 1$ . Hence  $F_i \simeq \mathbb{P}^{d_i}$  by [CMSB02].

Let  $H_i$  be the family of rational curves on  $X$  containing a point parametrizing lines in  $F_i \simeq \mathbb{P}^{d_i}$ . Then  $H_i$  is a dominating and unsplit family of rational curves since  $(-K_X \cdot \text{Fam } H_i) = d_i + 1 = l(R_i)$ .

We consider  $\mathcal{C}_x^i \subset \mathbb{P}(T_X|_x^\vee)$  for  $x \in F_i$ , which is a VMRT of  $H_i$  at  $x$ . We have  $\mathcal{C}_x^i = \mathbb{P}(T_{F_i}|_x^\vee) \subset \mathbb{P}(T_X|_x^\vee)$ ; a linear subspace of dimension  $d_i - 1$ . Hence  $X \simeq \prod_{i=1}^k \mathbb{P}^{d_i}$  by Theorem 2.10.  $\square$

**Proposition 2.12.** *Let  $X$  be a smooth projective variety of dimension  $n$ . If there exist distinct  $K_X$ -negative extremal rays  $R_1, R_2 \subset \overline{\text{NE}}(X)$  such that the intersection  $\text{Exc}(R_1) \cap \text{Exc}(R_2)$  is not empty. Then we have*

$$(l(R_1) - 1) + (l(R_2) - 1) \leq n$$

and equality holds if and only if  $X \simeq \mathbb{P}^{l(R_1)-1} \times \mathbb{P}^{l(R_2)-1}$ .

*Proof.* We fix an arbitrary point  $x \in \text{Exc}(R_1) \cap \text{Exc}(R_2)$ . For  $i = 1, 2$ , we set  $\varphi_i := \text{cont}_{R_i}: X \rightarrow Y_i$ ,  $y_i := \varphi_i(x) \in Y_i$ . Let  $C_i \subset X$  be a rational curve with

- (1)  $x \in C_i$  and  $[C_i] \in R_i$ ,
- (2)  $(-K_X \cdot C_i)$  is minimal among satisfying (1).

We also let  $H_i$  be a family of rational curves containing  $[C_i] \in \text{RatCurves}^n(X)$ . Then  $(H_i)_x$  is projective by construction. Hence we have

$$\begin{aligned} \dim \varphi_i^{-1}(y_i) &\geq \dim \text{Locus}(H_i)_x \\ &\geq (n - \dim \text{Locus}(H_i)) + (-K_X \cdot \text{Fam } H_i) - 1 \\ &\geq (-K_X \cdot \text{Fam } H_i) - 1 \geq l(R_i) - 1 \end{aligned}$$

by Proposition 2.2. We note that  $\dim(\varphi_1^{-1}(y_1) \cap \varphi_2^{-1}(y_2)) = 0$ . Indeed, if there exists an irreducible curve  $l \subset \varphi_1^{-1}(y_1) \cap \varphi_2^{-1}(y_2)$ , then  $[l] \in R_1 \cap R_2 = \{0\}$ , which leads to a contradiction. Thus we have  $n \geq \dim \varphi_1^{-1}(y_1) + \dim \varphi_2^{-1}(y_2)$ . Hence  $n \geq (l(R_1) - 1) + (l(R_2) - 1)$ .

If the equality holds, then  $H_i$  are dominating and unsplit for all  $i = 1, 2$  since  $(-K_X \cdot \text{Fam } H_i) = l(R_i)$  and  $\dim \text{Locus}(H_i) = n$ . Therefore one has  $X \simeq \mathbb{P}^{l(R_1)-1} \times \mathbb{P}^{l(R_2)-1}$  by [Occ06, Theorem 1.1].  $\square$

**Corollary 2.13.** *Let  $X$  be a Fano manifold of dimension  $n$  and Picard number 2. Then  $\text{NE}(X)$  is spanned by two extremal rays,  $R_1$  and  $R_2$ . If none of  $R_1$  and  $R_2$  is small, then we have*

$$(l(R_1) - 1) + (l(R_2) - 1) \leq n,$$

and the equality holds if and only if  $X \simeq \mathbb{P}^{l(R_1)-1} \times \mathbb{P}^{l(R_2)-1}$ .

*Proof.* For  $i = 1, 2$ , we set  $\varphi_i := \text{cont}_{R_i} : X \rightarrow Y_i$  and  $E_i := \text{Exc}(R_i)$ . Let  $C_i \subset X$  be an irreducible curve with  $[C_i] \in R_i$ . It is enough to show  $E_1 \cap E_2 \neq \emptyset$  by Proposition 2.12. We can assume  $R_1$  is divisorial. We can write  $K_X = \varphi_1^* K_{Y_1} + aE_1$  with  $a \in \mathbb{Q}_{>0}$  since  $Y_1$  has terminal singularities. We have  $(E_1 \cdot C_1) < 0$  since  $(K_X \cdot C_1) < 0$  and  $(\varphi_1^* K_{Y_1} \cdot C_1) = 0$ . Thus we have  $(E_1 \cdot C_2) > 0$  since  $E_1$  is a prime divisor and  $R_1 = \mathbb{R}_{\geq 0}[C_1]$  and  $R_2 = \mathbb{R}_{\geq 0}[C_2]$  span  $\text{NE}(X)$ . Therefore  $E_1 \cap C_2 \neq \emptyset$ . In particular  $E_1 \cap E_2 \neq \emptyset$  holds.  $\square$

### 3 Known classification results.

**Theorem 3.1** ([Cas09, Prop.3.1 and Thm.1.1]). *Let  $X$  be a Fano manifold of dimension  $n$  and Picard number  $\rho_X$ .*

- (1) *If  $n \geq 3$  and there exists  $R \subset \text{NE}(X)$  an extremal ray of type  $(n-1, 0)$ , then  $\rho_X \leq 3$ .*
- (2) *If  $n \geq 4$  and there exists  $R \subset \text{NE}(X)$  an extremal ray of type  $(n-1, 1)$ , then  $\rho_X \leq 5$ .*

**Theorem 3.2** ([AO02, Theorem 5.1]). *Let  $X$  be a smooth projective variety of dimension  $n$  and  $R \subset \overline{\text{NE}}(X)$  be a  $K_X$ -negative extremal ray of type  $(n-1, m)$  such that:*

- $l(R) \geq n-1-m$ ,



- all non trivial fibers of  $\text{cont}_R$  are equi-dimensional.

Then  $R$  is of type  $(n-1, m)^{\text{sm}}$ .

**Proposition 3.3** ([Tsu10b, Proposition 5] (and [AO02, Theorem 5.1])). *Let  $X$  be a Fano manifold of dimension  $n \geq 4$ . Assume that there exist distinct extremal rays  $R_1, R_2 \subset \text{NE}(X)$  such that  $R_i$  are of type  $(n-1, 1)$  and  $l(R_i) = n-2$  for  $i = 1$  and  $2$ . Then  $\text{Exc}(R_1) \cap \text{Exc}(R_2) = \emptyset$ .*

**Theorem 3.4** ([BCW02, Theorem 1.1]). *Let  $Y$  be a smooth projective variety of dimension  $n \geq 3$  and  $a \in Y$  be a (closed) point. Then  $X := \text{Bl}_a(Y)$  is a Fano manifold if and only if one of the following holds:*

- (i)  $Y \simeq \mathbb{P}^n$  and  $a \in Y$  is an arbitrary point.
- (ii)  $Y \simeq \mathbb{Q}^n$  and  $a \in Y$  is an arbitrary point.
- (iii)  $Y \simeq V_d$  with  $1 \leq d \leq n$  and  $a \notin H'$  (the strict transform of  $H$ ) with  $V_d := \text{Bl}_Z(\mathbb{P}^n)$ , where  $H \subset \mathbb{P}^n$  is a hyperplane and  $Z \subset H$  is a smooth subvariety of dimension  $n-2$  and degree  $d$ .

**Remark 3.5.** The examples in Theorem 3.4, we have the following properties by easy calculations.

- (i) If  $X = \text{Bl}_a(Y)$  is in Theorem 3.4 (i), then

$$\begin{aligned} \text{NE}(X) &= \mathbb{R}_{\geq 0}[f] + \mathbb{R}_{\geq 0}[g], \\ (-K_X \cdot f) &= 2, \\ (-K_X \cdot g) &= n-1 \end{aligned}$$

holds, where  $f$  is the strict transform of a line on  $Y = \mathbb{P}^n$  passing through  $a$  and  $g$  is a line in the exceptional divisor ( $\simeq \mathbb{P}^{n-1}$ ) of  $X \rightarrow Y$ . Thus  $s(X) = n-1$ .

- (ii) If  $X = \text{Bl}_a(Y)$  is in Theorem 3.4 (ii), then

$$\begin{aligned} \text{NE}(X) &= \mathbb{R}_{\geq 0}[f] + \mathbb{R}_{\geq 0}[g], \\ (-K_X \cdot f) &= 1, \\ (-K_X \cdot g) &= n-1 \end{aligned}$$

holds, where  $f$  is the strict transform of a line on  $Y = \mathbb{Q}^n$  passing through  $a$  and  $g$  is a line in the exceptional divisor ( $\simeq \mathbb{P}^{n-1}$ ) of  $X \rightarrow Y$ . Thus  $s(X) = n-2$ .

- (iii) If  $X = \text{Bl}_a(Y)$  is in Theorem 3.4 (iii), then

$$\begin{aligned} \text{NE}(X) &= \mathbb{R}_{\geq 0}[f] + \mathbb{R}_{\geq 0}[g] + \mathbb{R}_{\geq 0}[l] + \mathbb{R}_{\geq 0}[m], \\ l &\equiv m + g + (1-d)f \text{ in } N_1(X), \\ (-K_X \cdot f) &= 1, \quad (-K_X \cdot g) = 1, \\ (-K_X \cdot l) &= n+1-d, \quad (-K_X \cdot m) = 1 \end{aligned}$$

holds, where  $f \subset X$  is a fiber over  $Z$ ,  $g \subset X$  is a line in a fiber over  $a$ ,  $l \subset X$  is a line in  $H'$ , and  $m \subset X$  is a strict transform of a line passing through  $a$  and a point in  $Z$ . Thus if  $d = 1$  then  $s(X) = n - 2$ , but if  $d > 1$  then  $s(X) = 2n - 2 - d$ .

**Theorem 3.6** ([Tsu10a, Theorem 1], [Tsu10c, Propositions 3, 4]). *Let  $Y$  be a smooth projective variety of dimension  $n \geq 4$ ,  $C \subset Y$  be a smooth curve, and  $X := \text{Bl}_C(Y)$ . We assume that  $X$  is a Fano manifold of Picard number  $\rho_X$ .*

- (1) *If  $\rho_X = 5$ , then one of the following holds:*
  - (i)  $Y \simeq \text{Bl}_{\{p\} \cup \{q\} \cup \mathbb{P}^{n-2}}(\mathbb{P}^n)$  with  $\mathbb{P}^{n-2} \cap \overline{pq} = \emptyset$  and  $C$  is the strict transform of  $\overline{pq}$ , where  $\overline{pq} \subset \mathbb{P}^n$  is the line through  $p, q$ .
  - (ii)  $Y \simeq \text{Bl}_{\{p\} \cup \{q\} \cup \mathbb{Q}^{n-2}}(\mathbb{P}^n)$  with  $\mathbb{Q}^{n-2} \cap \overline{pq} = \emptyset$  and  $C$  is the strict transform of  $\overline{pq}$ , where  $\overline{pq} \subset \mathbb{P}^n$  is the line through  $p$  and  $q$ .
- (2) *Assume that there exists an extremal ray  $R \subset \text{NE}(X)$  of fiber type with  $l(R) \geq 2$ .*
  - *If  $R$  is of type  $(n, n - 2)$ , then  $\rho_X = 2$ .*
  - *If  $R$  is of type  $(n, n - 1)$ , then the pair of  $(Y, C)$  is one of the following:*
    - (i)  $Y \simeq \mathbb{Q}^n$  and  $C$  is a line in  $\mathbb{Q}^n \subset \mathbb{P}^{n+1}$ .
    - (ii)  $Y \simeq \mathbb{P}^1 \times \mathbb{P}^{n-1}$  and  $C$  is a fiber of the second projection.
    - (iii)  $Y \simeq \text{Bl}_{\mathbb{P}^{n-2}}(\mathbb{P}^n)$  and  $C$  is the strict transform of a line in  $\mathbb{P}^n$  disjoint from  $\mathbb{P}^{n-2}$ .
    - (iv)  $Y \simeq \text{Bl}_{\mathbb{P}^{n-2}}(\mathbb{P}^n)$  and  $C$  is a fiber of the blowing up.
    - (v)  $Y \simeq \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n-1})$  and  $C$  is the section of  $\mathbb{P}^{n-1}$ -bundle over  $\mathbb{P}^1$  whose normal bundle  $\mathcal{N}_{C/Y}$  is isomorphic to  $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus n-1}$ .

**Remark 3.7.** The examples in Theorem 3.4, we have the following properties by easy calculations.

- (1) (i) If  $X = \text{Bl}_C(Y)$  is in Theorem 3.6 (1i), then

$$\begin{aligned}
\text{NE}(X) &= \mathbb{R}_{\geq 0}[e] + \mathbb{R}_{\geq 0}[f] + \mathbb{R}_{\geq 0}[g] + \mathbb{R}_{\geq 0}[h] \\
&\quad + \mathbb{R}_{\geq 0}[k] + \mathbb{R}_{\geq 0}[l] + \mathbb{R}_{\geq 0}[m] \\
(-K_X \cdot e) &= n - 2, \quad (-K_X \cdot f) = 1, \quad (-K_X \cdot g) = 1, \\
(-K_X \cdot h) &= 1, \quad (-K_X \cdot k) = 1, \quad (-K_X \cdot l) = 1, \quad (-K_X \cdot m) = 1,
\end{aligned}$$

and  $\text{NE}(X)$  is exactly spanned by those seven extremal rays, where

- $e$  is a nontrivial fiber of the morphism  $X \rightarrow Y$ ,
- $f$  is the strict transform of a line ( $\simeq \mathbb{P}^{n-1} \subset Y$ ) in the exceptional divisor over  $p$ ,
- $g$  is the strict transform of a line ( $\simeq \mathbb{P}^{n-1} \subset Y$ ) in the exceptional divisor over  $q$ ,

- $h$  is a fiber over  $\mathbb{P}^{n-2}$ ,
- $k$  is a fiber of  $E \simeq C \times \mathbb{P}^{n-2} \rightarrow \mathbb{P}^{n-2}$ , where  $E$  is the exceptional divisor of  $X \rightarrow Y$ ,
- $l$  is the strict transform of a line in  $\mathbb{P}^n$  passing through  $p$  and  $\mathbb{P}^{n-2}$ ,
- $m$  is the strict transform of a line in  $\mathbb{P}^n$  passing through  $q$  and  $\mathbb{P}^{n-2}$ .

Thus  $s(X) = n - 3$ .

(ii) If  $X = \text{Bl}_C(Y)$  is in Theorem 3.6 (1ii), then

$$\begin{aligned} \text{NE}(X) &= \mathbb{R}_{\geq 0}[e] + \mathbb{R}_{\geq 0}[f] + \mathbb{R}_{\geq 0}[g] + \mathbb{R}_{\geq 0}[h] \\ &\quad + \mathbb{R}_{\geq 0}[j] + \mathbb{R}_{\geq 0}[k] + \mathbb{R}_{\geq 0}[l] + \mathbb{R}_{\geq 0}[m] \\ (-K_X \cdot e) &= n - 2, \quad (-K_X \cdot f) = 1, \quad (-K_X \cdot g) = 1, \quad (-K_X \cdot h) = 1, \\ (-K_X \cdot j) &= 1, \quad (-K_X \cdot k) = 1, \quad (-K_X \cdot l) = 1, \quad (-K_X \cdot m) = 1, \end{aligned}$$

and  $\text{NE}(X)$  is exactly spanned by those eight extremal rays, where

- $e$  is a nontrivial fiber of the morphism  $X \rightarrow Y$ ,
- $f$  is the strict transform of a line ( $\simeq \mathbb{P}^{n-1} \subset Y$ ) in the exceptional divisor over  $p$ ,
- $g$  is the strict transform of a line ( $\simeq \mathbb{P}^{n-1} \subset Y$ ) in the exceptional divisor over  $q$ ,
- $h$  is a fiber over  $\mathbb{Q}^{n-2}$ ,
- $j$  is the strict transform of a line in  $\mathbb{P}^n$  intersects  $\overline{pq}$  with each other and is contained in a unique hyperplane in  $\mathbb{P}^n$  which contains  $\mathbb{Q}^{n-2}$ ,
- $k$  is a fiber of  $E \simeq C \times \mathbb{P}^{n-2} \rightarrow \mathbb{P}^{n-2}$ , where  $E$  is the exceptional divisor of  $X \rightarrow Y$ ,
- $l$  is the strict transform of a line in  $\mathbb{P}^n$  passing through  $p$  and  $\mathbb{Q}^{n-2}$ ,
- $m$  is the strict transform of a line in  $\mathbb{P}^n$  passing through  $q$  and  $\mathbb{Q}^{n-2}$ .

Thus  $s(X) = n - 3$ .

- (2) (i) If  $X = \text{Bl}_C(Y)$  is in Theorem 3.6 (2i), then  $\rho_X = 2$ . Thus  $s(X) < n$  by Corollary 2.13.
- (ii) If  $X = \text{Bl}_C(Y)$  is in Theorem 3.6 (2ii), then

$$\begin{aligned} \text{NE}(X) &= \mathbb{R}_{\geq 0}[f] + \mathbb{R}_{\geq 0}[g] + \mathbb{R}_{\geq 0}[h] \\ (-K_X \cdot f) &= n - 2, \quad (-K_X \cdot g) = 2, \quad (-K_X \cdot h) = 2 \end{aligned}$$

holds, where  $f$  is a nontrivial fiber of  $X \rightarrow Y$ ,  $g$  is the strict transform of a general fiber of the first projection  $Y = \mathbb{P}^1 \times \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{n-1}$  and  $h$  is the strict transform of a line in the second projection  $Y = \mathbb{P}^1 \times \mathbb{P}^{n-1} \rightarrow \mathbb{P}^1$  passing through  $C$ . Thus  $s(X) = n - 1$ .

(iii) If  $X = \text{Bl}_C(Y)$  is in Theorem 3.6 (2iii), then

$$\begin{aligned} \text{NE}(X) &= \mathbb{R}_{\geq 0}[f] + \mathbb{R}_{\geq 0}[g] + \mathbb{R}_{\geq 0}[h] \\ (-K_X \cdot f) &= n - 2, \quad (-K_X \cdot g) = 1, \quad (-K_X \cdot h) = 2 \end{aligned}$$

holds, where  $f$  is a nontrivial fiber of  $X \rightarrow Y$ ,  $g$  is a fiber over  $\mathbb{P}^{n-2}$  and  $h$  is the strict transform of a line in  $\mathbb{P}^n$  passing through  $C$  and  $\mathbb{P}^{n-2}$ . Thus  $s(X) = n - 2$ .

(iv) If  $X = \text{Bl}_C(Y)$  is in Theorem 3.6 (2iv), then

$$\begin{aligned} \text{NE}(X) &= \mathbb{R}_{\geq 0}[f] + \mathbb{R}_{\geq 0}[g] + \mathbb{R}_{\geq 0}[h] \\ (-K_X \cdot f) &= n - 2, \quad (-K_X \cdot g) = 1, \quad (-K_X \cdot h) = 2 \end{aligned}$$

holds, where  $f$  is a nontrivial fiber of  $X \rightarrow Y$ ,  $g$  is a general fiber over  $\mathbb{P}^{n-2}$  and  $h$  is the strict transform of a line in  $\mathbb{P}^n$  passing through  $\mathbb{P}^{n-2}$  and the image of  $C$  in  $\mathbb{P}^n$ . Thus  $s(X) = n - 2$ .

(v) If  $X = \text{Bl}_C(Y)$  is in Theorem 3.6 (2v), then

$$\begin{aligned} \text{NE}(X) &= \mathbb{R}_{\geq 0}[f] + \mathbb{R}_{\geq 0}[g] + \mathbb{R}_{\geq 0}[h] \\ (-K_X \cdot f) &= n - 2, \quad (-K_X \cdot g) = 1, \quad (-K_X \cdot h) = 2 \end{aligned}$$

holds, where  $f$  is a nontrivial fiber of  $X \rightarrow Y$ ,  $g$  is a fiber of  $E \simeq C \times \mathbb{P}^{n-2} \rightarrow \mathbb{P}^{n-2}$ , where  $E$  is the exceptional divisor of  $X \rightarrow Y$ , and  $h$  is the strict transform of a line in a fiber of  $Y \rightarrow \mathbb{P}^1$  passing through  $C$ . Thus  $s(X) = n - 2$ .

## 4 Proof of Theorem 1.5

If  $\rho_X = 1$ , then  $s(X) \leq n$  holds and the equality holds if and only if  $X \simeq \mathbb{P}^n$  by [CMSB02]. Hence we can assume  $\rho_X \geq 2$ .

### 4.1 Proof of Theorem 1.5 (i)

We can assume  $n = 3$  since the case  $n \leq 2$  is trivial. We prove the assertion without using the result [MM81] of complete classification of Fano 3-folds with  $\rho_X \geq 2$ .

If  $\rho_X = 2$  then the assertion holds by Corollary 2.13. We can assume  $\rho_X \geq 3$ . By Theorem 2.6, Remark 2.7 (1) and Theorem 3.2, if there exists an extremal ray  $R \subset \text{NE}(X)$  which satisfies that  $l(R) \geq 2$ , then one of the following holds:

(A)  $R$  is of type  $(2, 0)^{\text{sm}}$  and  $l(R) = 2$ .

(B)  $R$  is of type  $(3, 2)$  and  $l(R) = 2$ .

(We note that this can be obtained directly using complete classification of extremal contractions for 3-folds [Mor82, Theorems 3.3, 3.5].)

If there exists an extremal ray  $R \subset \text{NE}(X)$  of type (A), then  $X \simeq \text{Bl}_a(V_d)$  with  $1 \leq d \leq 3$  by Theorem 3.4, thus  $s(X) < 3$  by Remark 3.5 (iii). If there exist distinct extremal rays  $R_1, R_2, R_3 \subset \text{NE}(X)$  such that all of them are of type (B), then  $X \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  by Theorem 2.11.

Therefore we have completed the proof of Theorem 1.5 (i).

## 4.2 Proof of Theorem 1.5 (ii)

If  $\rho_X = 2$  then the assertion holds by Corollary 2.13. (We note that if  $\rho_X = 2$  and both extremal rays are small, then  $s(X) = 0$ .) We can assume  $\rho_X \geq 3$ . By Theorem 2.6, Remark 2.7 (1) and Theorem 3.2, if there exists an extremal ray  $R \subset \text{NE}(X)$  which satisfies that  $l(R) \geq 2$ , then one of the following holds:

- (A)  $R$  is of type  $(3, 0)^{\text{sm}}$  and  $l(R) = 3$ .
- (B)  $R$  is of type  $(3, 0)$  and  $l(R) = 2$ .
- (C)  $R$  is of type  $(3, 1)^{\text{sm}}$  and  $l(R) = 2$ .
- (D)  $R$  is of type  $(4, 3)$  and  $l(R) = 2$ .
- (E)  $R$  is of type  $(4, 2)$  and  $l(R) = 3$ .
- (F)  $R$  is of type  $(4, 2)$  and  $l(R) = 2$ .

If there exists an extremal ray  $R \subset \text{NE}(X)$  of type (A), then  $s(X) < 4$  excepts for  $X \simeq \text{Bl}_a(V_2) \simeq \text{Bl}_{p,q}(\mathbb{Q}^n)$ , and if  $X \simeq \text{Bl}_{p,q}(\mathbb{Q}^n)$  then  $s(X) = 4$  by Theorem 3.4 and Remark 3.5 (iii). Hence we can further assume that there exist distinct extremal rays  $R_1, \dots, R_m \subset \text{NE}(X)$  such that

- $\sum_{1 \leq i \leq m} (l(R_i) - 1) \geq 4$ ,
- $R_i$  are of type (B) or ... or (F) for all  $1 \leq i \leq m$

It is enough to show  $X \simeq \prod_{1 \leq i \leq m} \mathbb{P}^{l(R_i)-1}$  under above assumptions.

If  $R_i$  are of fiber type for all  $1 \leq i \leq m$ , then  $X \simeq \prod_{1 \leq i \leq m} \mathbb{P}^{l(R_i)-1}$  by Theorem 2.11. If  $R_i$  is of type (B), then  $R_j$  are of type (B) or (C) and  $\text{Exc}(R_i) \cap \text{Exc}(R_j) = \emptyset$  for all  $j \neq i$  by Proposition 2.5. If  $R_i$  is of type (C), then  $R_j$  are of type (B) or (C) and  $\text{Exc}(R_i) \cap \text{Exc}(R_j) = \emptyset$  for all  $j \neq i$ . Indeed, if  $R_j$  is of type (D), (E) or (F) then  $s(X) < 4$  by Theorem 3.6 (2) and Remark 3.7 (2), which leads to a contradiction for our additional assumptions. Furthermore, if  $R_j$  is of type (C), then  $\text{Exc}(R_i) \cap \text{Exc}(R_j) = \emptyset$  by Proposition 3.3.

Thus we can further assume such that

- $m = 4$ ,
- $R_i$  are of type (B) or (C) for all  $1 \leq i \leq 4$ ,
- $\text{Exc}(R_i) \cap \text{Exc}(R_j) = \emptyset$  for all  $1 \leq i < j \leq 4$

It is enough to show that there does not exist Fano 4-fold  $X$  satisfying above properties.

We have  $\rho_X \leq 5$  by Theorem 3.1 (2). However, if the equality holds then  $s(X) < 4$  by Theorem 3.6 (1) and Remark 3.7 (1). Hence  $\rho_X \leq 4$ . Since  $\text{Exc}(R_i) \cap \text{Exc}(R_j) = \emptyset$  for all  $1 \leq i < j \leq 4$ , we can define  $\varphi: X \rightarrow Z$  which is the gluing morphism of  $\text{cont}_{R_1}, \dots, \text{cont}_{R_4}$  contracting  $\text{Exc}(R_1), \dots, \text{Exc}(R_4)$ .

If  $R_i$  are of type (C) for all  $1 \leq i \leq 4$ , then  $Z$  is a smooth proper variety of  $\rho_Z = \rho_X - 4 \leq 0$ , hence this leads to a contradiction. If  $R_1$  is of type (B), then  $\rho_X \leq 3$  by Theorem 3.1 (1). However,  $Z$  is a normal  $\mathbb{Q}$ -factorial proper variety by Remark 2.7 (2) and there exists an exact sequence of the form

$$0 \longrightarrow N^1(Z) \xrightarrow{\varphi^*} N^1(X) \xrightarrow{((\bullet \cdot C_1), \dots, (\bullet \cdot C_4))} \mathbb{R}^{\oplus 4}.$$

Thus  $\rho_Z \leq \rho_X - 3 \leq 0$  since at least three points in  $\{[C_1], \dots, [C_4]\}$  are linearly independent in  $N_1(X)$ . This leads to a contradiction.

As a consequence, we have completed the proof of Theorem 1.5 ii.

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